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Perturbative Thermodynamics of Lattice QCD with Chiral-Invariant Four-Fermion Interactions

Michael Chavel*
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics
Massachusetts Institute of Technology, Cambridge, MA 02139-4307
and
Department of Physics
University of Illinois, Urbana, IL 61801-3080
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Abstract

Lattice QCD with additional chiral-invariant four-fermion interactions is studied at nonzero temperature. Staggered Kogut-Susskind quarks are used. The four-fermion interactions are implemented by introducing bosonic auxiliary fields. A mean field treatment of the auxiliary fields is used to calculate the model's asymptotic scale parameter and perturbative thermodynamics, including the one-loop gluonic contributions to the energy, entropy, and pressure. In this approach the calculations reduce to those of ordinary lattice QCD with massive quarks. Hence, the previous calculations of these quantities in lattice QCD using massless quarks are generalized to the massive case.

*Electronic address: chavel@uiuc.edu

I. INTRODUCTION

One of the remaining challenges within the standard model of particle physics is to understand the high temperature behavior of QCD with light quarks. This includes determining the location and order of the chiral phase transition, associated with the restoration of the $\mathrm{SU}(N_f)_{\mathrm{L}} \otimes \mathrm{SU}(N_f)_{\mathrm{R}}$ flavor symmetry, and whether it obeys the expected dimensional reduction scenario [1]. In the high temperature plasma phase, the status of the $\mathrm{U}(1)_{\mathrm{A}}$ symmetry, the equation of state, and the gluon screening masses, necessary for high temperature perturbation theory, still remain to be determined.

Lattice gauge theory appears to be an excellent nonperturbative tool for studying these issues. In fact, many aspects of the deconfining phase transition and the high temperature plasma phase in the gluonic sector are now well understood from Monte Carlo studies of pure SU(3) gauge theory [2]. Unfortunately, simulations of full QCD, including dynamical quarks, are much more computationally intensive and so must be performed on smaller lattices with relatively large lattice spacings and/or finite size effects. Ref. [3] contains some recent reviews.

In an effort to more closely approach the true physics of continuum QCD, many researchers have turned to using improved [4] or "perfect" [5] lattice actions, which reduce discretization errors and lattice artifacts. A somewhat different approach is the χQCD model [6–9]. In this model, extra auxiliary fields are introduced, which upon integration yield chiral invariant four-fermion interactions of the form $\frac{G}{2N_f/4}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\tau\psi)^2]$. Such interactions are irrelevant to the long distance physics and do not survive in the continuum limit. However, when using the hybrid Monte Carlo algorithm the auxiliary fields entering the fermion determinant allow one to avoid the usual zero mass singularity when inverting the Dirac operator and perform simulations directly in the chiral limit. (i. e. the bare quark mass can be set exactly to zero.)

To be specific, the χQCD action for N_f flavors of staggered quarks on an anisotropic lattice with spatial (temporal) lattice spacing a (a_τ) is

$$
\begin{aligned}
S = \sum_x \Big[\beta_\sigma \sum_{i<j} P_{ij}(x) + \beta_\tau \sum_j P_{0j}\Big] + \sum_{a=1}^{N_f/4} \sum_{x,y} a^2 a_\tau \bar{\chi}^a(x) Q(x,y) \chi^a(y) \\
+ \frac{N_f}{8}\gamma \sum_{\tilde{x}} a^3 a_\tau [\sigma^2(\tilde{x}) + \pi^2(\tilde{x})], \qquad (1)
\end{aligned}
$$

with[1]

$$
Q(x,y) = \sum_{j=1}^{3} \mathcal{M}_j(x,y) + \gamma_F \mathcal{M}_0(x,y) + \delta_{x,y} \frac{1}{16} \sum_{<x,\tilde{x}>} a[\sigma(\tilde{x}) + i\varepsilon(x)\pi(\tilde{x})] \qquad (2)
$$

[1] The model as considered here has a $\mathrm{U}(1)_{\mathrm{L}} \otimes \mathrm{U}(1)_{\mathrm{R}}$ global symmetry, generated by 1 and $\gamma_5\tau_3$ (τ_3 being the flavor equivalent of γ_5). One can also consider more physically realistic $\mathrm{SU}(N_f)_{\mathrm{L}} \otimes \mathrm{SU}(N_f)_{\mathrm{R}}$ symmetries. However, for the calculations considered in this paper, only the total number of quark flavors is relevant.

and $\gamma = 1/G$. The staggered hopping term is

$$\mathcal{M}_\nu(x,y) = \frac{1}{2}\eta_\nu(x)[U_\nu(x)\delta_{y,x+\hat{\nu}} - U_\nu(x-\hat{\nu})\delta_{y,x-\hat{\nu}}], \tag{3}$$

where $\eta_\nu(x) \equiv (-1)^{x_0+\cdots+x_{\nu-1}}$. The auxiliary fields $\sigma(\tilde{x})$ and $\pi(\tilde{x})$ are defined on the dual lattice sites. The symbol $< \tilde{x}, x >$ represents the 16 dual sites $\tilde{x}$ adjacent to the direct lattice site x, and $\varepsilon(x)$ is the alternating phase $(-1)^{x_0+x_1+x_2+x_3}$. See [10] for a more detailed discussion of the auxiliary field formulation used here. $P_{\mu\nu}$ denotes the conventional Wilson plaquettes:

$$P_{\mu\nu} \equiv 1 - \frac{1}{N}\mathrm{ReTr}[U_{x,x+\mu}U_{x+\mu,x+\mu+\nu}U^\dagger_{x+\nu,x+\mu+\nu}U^\dagger_{x,x+\nu}]. \tag{4}$$

The separate gauge couplings $\beta_\sigma = 6/g_\sigma^2$, $\beta_\tau = 6/g_\tau^2$ as well as the extra parameter γ_F are introduced in the usual way to maintain Euclidean invariance in the continuum limit, when studying the model on anisotropic lattices at nonzero temperature.

This paper studies the asymptotic scaling of the χQCD model and its perturbative thermodynamics. Section II studies how the four-fermion coupling affects the approach to the continuum limit and the lattice scale parameter Λ_L. Section III deals with the perturbative thermodynamics of the model and presents the one-loop gluonic corrections to the energy, entropy, and pressure. Throughout the paper, the four-fermion interactions are dealt with via a mean field (or, equivalently, $N_f \to \infty$) approximation, the details of which are given in Sec. IV. Treating the model in this way, the effects of the four-fermion interactions on the asymptotic scaling behavior and the perturbative thermodynamics are governed solely by the value of the dynamical quark mass $m^2 = \langle\sigma\rangle^2 + \langle\pi\rangle^2$. Therefore, the one-loop gluonic corrections to the model's thermodynamic observables reduce to those of standard lattice QCD with a nonzero bare quark mass. Since such calculations have only appeared in the literature before at zero quark mass [11–13], they are extended to nonzero masses in Sec. V.

II. χQCD SCALE PARAMETER

Consider the χQCD model on an isotropic lattice ($a_\tau = a$), with $\beta_\sigma = \beta_\tau \equiv 6/g^2$ and $\gamma_F = 1$. In the asymptotic scaling region of lattice QCD the gauge coupling g is related to the lattice spacing a by

$$a\Lambda_\mathrm{L} = (\beta_0 g^2)^{-\beta_1/2\beta_0^2}\exp\Big(-\frac{1}{2\beta_0 g^2}\Big)\Big[1 + O(g^2)\Big], \tag{5}$$

which defines the conventional lattice QCD scale parameter Λ_L. In the continuum limit the scale parameter is the only dimensional parameter. Hence, all physical quantities measured on the lattice (such as particle masses, string tension, etc.) must be expressible in terms of Λ_L by simple dimensional analysis.

The numerical value of Λ_L depends upon the specifics of the lattice regularization, and so is not universal. The ratio of the lattice and continuum scale parameters, in any two specific regularization schemes, is given by

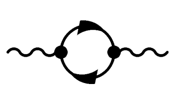
$$\frac{\Lambda_{\rm L}}{\Lambda_{\rm c}} = Ma \exp\Big[-\frac{1}{2\beta_0}\Big(\frac{1}{g^2} - \frac{1}{g_{\rm c}^2}\Big)\Big], \tag{6}$$

where M is the continuum renormalization mass scale of interest (momentum subtraction point or Pauli-Villars mass, etc.). This ratio is most easily calculated by a one-loop background field calculation [14], where the gluonic action is expanded around a slowly varying classical background configuration.

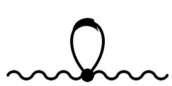
In χQCD one has the additional four-fermion coupling G, which has physical dimensions of (length)^2. In the continuum limit such an interaction is perturbatively irrelevant. Therefore, the asymptotic scaling formula (5) must remain valid in some window of $a \to 0$ for any given value of G.[2] The numerical value of the scale parameter will change depending on the particular value of G used. However, this dependence must cancel out when any physical quantities are expressed in terms of the continuum QCD scale parameter. This will need to be verified in future Monte Carlo simulations.

The effects of the explicit four-fermion interaction in χQCD are most easily observed by a mean field calculation, or a large flavor expansion, where the induced dynamical quark mass is related to the mean values of the auxiliary fields (see Sec. IV). The calculation of the lattice scale parameter in lattice perturbation theory is then only affected by the presence of this dynamical mass in the fermionic contributions to the gluon self-energy (vacuum polarization).

The relevant diagrams are:



$$= \frac{N_f}{4}\frac{1}{4}\int_{q/2} \mathrm{Tr}[\gamma_\mu S_F(q-k/2)\gamma_\nu S_F(q+k/2)] \cos(q_\mu a_\mu)\cos(q_\nu a_\nu), \tag{7}$$



$$= -\frac{N_f}{4}\frac{1}{4}\int_{q/2} \mathrm{Tr}[\gamma_\mu S_F(q)] i a_\mu \sin(q_\mu a_\mu)\delta_{\mu\nu}. \tag{8}$$

(The notation is explained in the appendix.) Following [15,16] and collecting together the gauge, ghost, and fermion contributions, the ratio of the χQCD scale parameter Λ_χ to the scale parameter $\Lambda_{\rm MIN}$ in the minimal subtraction scheme can be written

$$\frac{\Lambda_\chi}{\Lambda_{\rm MIN}} = \exp\Big[J + \frac{1}{\beta_0}\Big(\frac{1}{48} - 3P + \frac{N_f}{2}P_5\Big)\Big], \tag{9}$$

with

---

[2] Unless, of course, their exists a second order phase transition at some value of $G \equiv G_c$, where the four-fermion interaction becomes both renormalizable and nontrivial. In this case, Eq. (5) may only hold when G is taken smaller than G_c.

$$J = \tfrac{1}{2}(\ln 4\pi - \gamma_{\rm Euler}) = 0.9769042, \tag{10}$$

$$P = 0.0849780, \tag{11}$$

$$P_5 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^4q}{(2\pi)^4} \frac{\cos^2(q_1)\cos^2(q_2) - \frac{1}{3}\cos(2q_1)\cos(2q_2)}{[\Delta_2(\xi) + m^2a^2]^2} - \frac{2}{3}\int_{-\infty}^{\infty} \frac{d^4q}{(2\pi)^4}\Big[\frac{1}{(q^2+m^2a^2)^2} - \frac{1}{(q^2+1)^2}\Big]. \tag{12}$$

Here, $m^2 = \langle\sigma\rangle^2 + \langle\pi\rangle^2$ is a function of the two bare couplings $\{G, g\}$. In simulations of χQCD the value of the four-fermion coupling is to be chosen very small, such that the observed values of $\langle\sigma\rangle$ and $\langle\pi\rangle$ are principally due to *nonperturbative effects of the gluons*. Therefore, they cannot be reliably calculated in perturbation theory and must be determined by Monte Carlo simulation. For this reason, it is best to keep Λ_χ parameterized by the dynamical quark mass $ma = a\sqrt{\langle\sigma\rangle^2 + \langle\pi\rangle^2}$.

Values of $\Lambda_{\rm MIN}/\Lambda_\chi$ are plotted as a function of ma in Figs. 1 and 2 for $N_f = 2$ and $N_f = 4$, respectively. At $ma = 0$ ($G = 0$) I find

$$\frac{\Lambda_{\rm MIN}}{\Lambda_\chi} = \frac{\Lambda_{\rm MIN}}{\Lambda_{\rm L}} = \begin{cases} 10.846 & N_f = 0 \\ 16.518 & N_f = 2 \\ 28.779 & N_f = 4 \end{cases}, \tag{13}$$

in agreement with [16]. As ma increases the ratio increases monotonically. Assuming reasonable values such as $\langle\psi\bar{\psi}\rangle \leq 1$ and $\gamma \geq 10$, ma will be 0.1 or less. At $ma = 0.1$ the increase in $\Lambda_{\rm MIN}/\Lambda_\chi$ is about 0.15% (0.3%) for 2 (4) fermion flavors. Thus, it is a relatively small effect which will be difficult to extract from simulation data.

III. χQCD THERMODYNAMICS

Working in the canonical ensemble, the Helmholtz free energy $F = -T\ln Z$, where $Z = \mathrm{Tr}\exp(-S/T)$ is the lattice regularized partition function. The temperature $T = 1/a_\tau N_\tau$, where N_τ (N_σ) is the number of lattice sites in the temporal (spatial) direction(s). Following the usual convention define $\xi \equiv a/a_\tau$. The energy density and pressure are then given by

$$\epsilon = -\frac{1}{V}\frac{\partial \ln Z}{\partial T^{-1}} = \frac{T}{(aN_\sigma)^3}\Big\langle a_\tau \frac{\partial S}{\partial a_\tau}\Big\rangle = -T^4\Big(\frac{N_\tau}{\xi N_\sigma}\Big)^3\Big\langle \xi\frac{\partial S}{\partial \xi}\Big\rangle, \tag{14}$$

$$p = T\frac{\partial \ln Z}{\partial V} = -\frac{1}{3}\frac{T}{(aN_\sigma)^3}\Big\langle a\frac{\partial S}{\partial a}\Big\rangle = -\frac{T^4}{3}\Big(\frac{N_\tau}{\xi N_\sigma}\Big)^3\Big[\Big\langle \xi\frac{\partial S}{\partial \xi}\Big\rangle + \Big\langle a\frac{\partial S}{\partial a}\Big\rangle\Big]. \tag{15}$$

Using the thermodynamic relations $F = U - TS$ and $p = -(\partial F/\partial V)_T$ the entropy density can be expressed as

$$s = \frac{\epsilon + p}{T} = \frac{4}{3}\frac{\epsilon}{T} - \frac{T^3}{3}\Big(\frac{N_\tau}{\xi N_\sigma}\Big)^3\Big\langle a\frac{\partial S}{\partial a}\Big\rangle. \tag{16}$$

The deviation from ideal gas behavior is typically characterized by

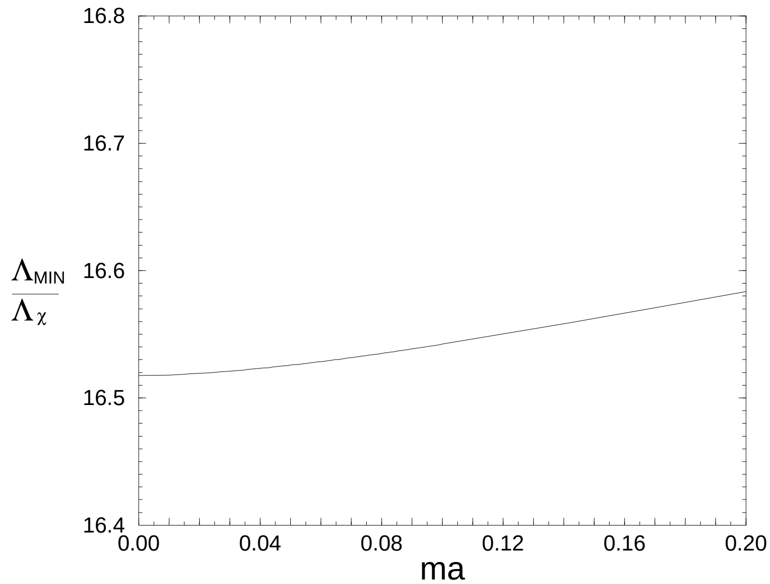


FIG. 1. $\frac{\Lambda_{\rm MIN}}{\Lambda_\chi}$ as a function of ma with $N_F = 2$.

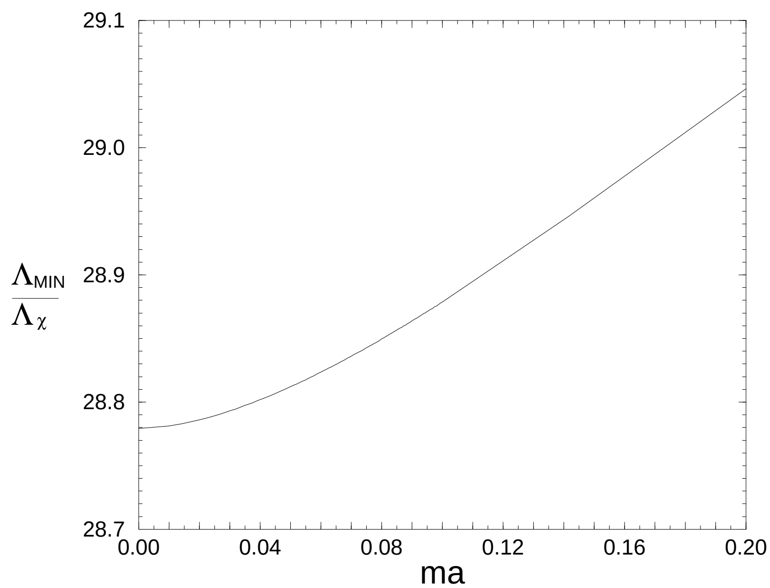


FIG. 2. $\frac{\Lambda_{\rm MIN}}{\Lambda_\chi}$ as a function of ma with $N_F = 4$.

$$\delta \equiv \epsilon - 3p = T^4\Big(\frac{N_\tau}{\xi N_\sigma}\Big)^3\Big\langle a\frac{\partial S}{\partial a}\Big\rangle. \tag{17}$$

Before proceeding any further, note that Gaussian integration over the auxiliary fields appearing in Eq. (1) will yield a factor of $(8\pi\xi/N_f\gamma a^4)^{N_\sigma N_\tau}$ in addition to the desired four-fermion interactions. Since the thermodynamic observables are to be computed by differentiating the partition function with respect to a and ξ, it is best to remove this factor from Z. This can be accomplished by adding to the action the extra term

$$\gamma\frac{N_f}{8}\sum_{\tilde{x}} a^3 a_\tau \bar{\eta}(\tilde{x})\eta(\tilde{x}), \tag{18}$$

where $\bar{\eta}$ and η free Grassmannian fields. Integration over $\bar{\eta}$ and η will then cancel the cancel the corresponding contribution from the auxiliary fields. Of course, if one is only concerned with measuring quantities relative to the zero temperature vacuum this becomes unnecessary.

Adding (18) to the χQCD action (1) it follows immediately that

$$\Big\langle\xi\frac{\partial S}{\partial\xi}\Big\rangle = N_\sigma^3 N_\tau\Big[\xi\frac{\partial\beta_\sigma}{\partial\xi}\langle P_{ss}\rangle + \xi\frac{\partial\beta_\tau}{\partial\xi}\langle P_{st}\rangle + \Big(-1+\xi\frac{\partial Z_\psi}{\partial\xi}\Big)\frac{N_f}{4}\langle\bar{\chi}Q\chi\rangle$$
$$+\xi\frac{\partial\gamma_F}{\partial\xi}\frac{N_f}{4}\langle\bar{\chi}\mathcal{M}_0\chi\rangle + \Big(-\gamma+\xi\frac{\partial\gamma}{\partial\xi}\Big)\frac{N_f}{8}\Big(\langle\sigma^2+\pi^2\rangle + \langle\bar{\eta}\eta\rangle\Big)\Big]. \tag{19}$$

All the averages on the right hand side are per space-time volume. $\langle P_{ss}\rangle$ ($\langle P_{st}\rangle$) stands for the average value of the space-space (space-time) plaquettes. Note that $\langle\bar{\chi}Q\chi\rangle = -3$ and $\gamma\frac{N_f}{8}\langle\bar{\eta}\eta\rangle = -1$.

In order to calculate the regularization dependence of the bare parameters appearing above, a mean field treatment of the auxiliary fields together with perturbative QCD corrections will be used. The details are presented in Secs. IV and V. In this approach the calculations of $\partial\beta_\sigma/\partial\xi$, $\partial\beta_\tau/\partial\xi$, and $\partial\gamma_F/\partial\xi$ become the same as in ordinary lattice QCD with $\langle\sigma\rangle^2+\langle\pi\rangle^2$ substituting for the square of the bare quark mass. The quantity $\partial\gamma/\partial\xi$ is determined by absorbing the ξ dependence of the one-loop gluonic corrections to the $\sigma\bar{\psi}\psi$ vertex into the value of $G = 1/\gamma$. This insures that the dynamical fermion mass, given by the pole in the quark propagator, is invariant with respect to ξ.

In Sec. IV it is shown that

$$\frac{1}{\gamma}\Big(\xi\frac{\partial\gamma}{\partial\xi}\Big)_{\xi=1} = -2g^2\Big(\frac{\partial C_m}{\partial\xi}\Big)_{\xi=1}. \tag{20}$$

The quantity $(\partial C_m/\partial\xi)_{\xi=1}$ is plotted in Fig. 10 as a function of $ma = a\sqrt{\langle\sigma\rangle^2+\langle\pi\rangle^2}$. It is the only quantity of interest which varies significantly with ma. [$(\partial C_m/\partial\xi)_{\xi=1}$ varies about 2% from $ma = 0$ to 0.1, while $\partial C_\sigma/\partial\xi$, $\partial C_\tau/\partial\xi$ and $\partial C_F/\partial\xi$ all vary 0.2% or less over the same range.] Hence, the values of $\partial\beta_\sigma/\partial\xi$, $\partial\beta_\tau/\partial\xi$ and $\partial\gamma_F/\partial\xi$ calculated with massless quarks can safely be substituted.

Differentiating the action with respect to the spatial lattice spacing yields

$$\left\langle a\frac{\partial S}{\partial a}\right\rangle = N_\sigma^3 N_\tau\Big[a\frac{\partial\beta_\sigma}{\partial a}\langle P_{ss}\rangle + a\frac{\partial\beta_\tau}{\partial a}\langle P_{st}\rangle + \Big(3+\frac{\partial Z_\psi}{\partial\ln a}\Big)\frac{N_f}{4}\langle\bar\chi Q\chi\rangle + a\frac{\partial\gamma_F}{\partial a}\frac{N_f}{4}\langle\bar\chi\mathcal{M}_0\chi\rangle$$
$$+\Big(4\gamma + a\frac{\partial\gamma}{\partial a}\Big)\frac{N_f}{8}\Big(\langle\sigma^2+\pi^2\rangle + \langle\bar\eta\eta\rangle\Big) + \frac{N_f}{4}\langle\bar\chi(\sigma+i\varepsilon\pi)\chi\rangle\Big]. \tag{21}$$

Using the identity $\frac{N_f}{4}\langle\bar\chi(\sigma+i\varepsilon\pi)\chi\rangle = 2(1-\gamma\frac{N_f}{8}\langle\sigma^2+\pi^2\rangle)$ the last two terms can be combined. To lowest order in g, the values of $\partial\beta_\sigma/\partial a$, $\partial\beta_\tau/\partial a$ and $\partial\gamma_F/\partial a$ are the same as in ordinary lattice QCD, since they are controlled by the lowest universal coefficient of the beta function (see Sec. V). There is no bare quark mass in the action. The expected scaling of such a mass is taken over here by the scaling of the four-fermion coupling. In Sec. IV it is shown that

$$\frac{1}{\gamma}\Big(a\frac{\partial\gamma}{\partial a}\Big)_{\xi=1} = -g^2\frac{2}{2\pi^2} + O(g^4). \tag{22}$$

Measuring the energy density and pressure with respect to the zero temperature vacuum, by subtracting measurements on a symmetric lattice (denoted by $\langle\cdots\rangle_{sym}$), the final expressions at $\xi=1$ are

$$\epsilon = T^4N_\tau^4\Big\{\frac{6}{g^2}\Big(1-g^2\frac{\partial C_\sigma}{\partial\xi}\Big)\Big[\langle P_{ss}\rangle - \langle P_{ss}\rangle_{sym}\Big] + \frac{6}{g^2}\Big(-1-g^2\frac{\partial C_\tau}{\partial\xi}\Big)\Big[\langle P_{st}\rangle - \langle P_{st}\rangle_{sym}\Big]$$
$$+\Big(1+g^2\frac{\partial C_F}{\partial\xi}\Big)\frac{N_f}{4}\Big[\langle\mathrm{Tr}\mathcal{M}_0Q^{-1}\rangle - \langle\mathrm{Tr}\mathcal{M}_0Q^{-1}\rangle_{sym}\Big]$$
$$+\Big(1+g^2 2\frac{\partial C_m}{\partial\xi}\Big)\gamma\frac{N_f}{8}\Big[\langle\sigma^2+\pi^2\rangle - \langle\sigma^2+\pi^2\rangle_{sym}\Big]\Big\}, \tag{23}$$
$$p = \frac{\epsilon}{3} + \frac{T^4}{3}N_\tau^4\Big\{12\beta_0\Big[\langle P_{ss}\rangle - \langle P_{ss}\rangle_{sym} + \langle P_{st}\rangle - \langle P_{st}\rangle_{sym}\Big]$$
$$+\Big(-2+g^2\frac{2}{2\pi^2}\Big)\gamma\frac{N_f}{8}\Big[\langle\sigma^2+\pi^2\rangle - \langle\sigma^2+\pi^2\rangle_{sym}\Big]\Big\}. \tag{24}$$

The (absolute) entropy density at $\xi=1$ is

$$s = \frac{4T^3}{3}N_\tau^4\Big\{\frac{6}{g^2}\Big[1+g^2\frac{1}{2}\Big(\frac{\partial C_\tau}{\partial\xi} - \frac{\partial C_\sigma}{\partial\xi}\Big)\Big]\Big[\langle P_{ss}\rangle - \langle P_{st}\rangle\Big]$$
$$+\Big(1+g^2\frac{\partial C_F}{\partial\xi}\Big)\frac{N_f}{4}\Big[\langle\mathrm{Tr}\mathcal{M}_0Q^{-1}\rangle - \frac{3}{4}\Big]$$
$$+\frac{1}{2}\Big(1+g^2\frac{1}{2\pi^2} + g^2 4\frac{\partial C_m}{\partial\xi}\Big)\Big[\gamma\frac{N_f}{8}\langle\sigma^2+\pi^2\rangle - 1\Big]\Big\}. \tag{25}$$

In writing the above expressions I have used the identity $\langle\bar\chi A\chi\rangle = -\langle\mathrm{Tr}AQ^{-1}\rangle$. The relations (66) and (82) from Sec. V have also been used to simplify the expression for the entropy density. For $ma = a\sqrt{\langle\sigma\rangle^2+\langle\pi\rangle^2}$ less than 0.1 the values of C_σ, C_τ, and C_F are effectively the same as in ordinary QCD:

$$\Big(\frac{\partial C_\sigma}{\partial\xi}\Big)_{\xi=1} = 0.20161 - \frac{N_f}{2}0.00062, \tag{26}$$

$$\left(\frac{\partial C_\tau}{\partial \xi}\right)_{\xi=1} = -0.13195 - \frac{N_f}{2}0.00782, \tag{27}$$

$$\left(\frac{\partial C_F}{\partial \xi}\right)_{\xi=1} = -0.2132. \tag{28}$$

They are discussed in full detail in Sec. V. Values of $(\partial C_m \partial \xi)_{\xi=1}$ are tabulated in Table I for values of $ma < 0.2$.

Note that $\gamma\frac{N_f}{8}\langle\sigma^2 + \pi^2\rangle$ approaches the Gaussian free field value of 1 as $\gamma \to \infty$. In which case, Eqs. (23) – (25) yield the correct expressions for massless QCD. Note also that in this model $\langle \mathrm{Tr} Q Q^{-1}\rangle = 3$ and $\frac{N_f}{4}\langle \mathrm{Tr}(\sigma + i\varepsilon\pi)Q^{-1}\rangle = 2(\gamma\frac{N_f}{8}\langle\sigma^2 + \pi^2\rangle - 1)$. Therefore,

$$\langle \mathrm{Tr}\mathcal{M}_0 Q^{-1}\rangle_{sym} = \tfrac{3}{4} - \tfrac{2}{N_f}(\gamma\tfrac{N_f}{8}\langle\sigma^2 + \pi^2\rangle_{sym} - 1). \tag{29}$$

As of this writing the only numerical study of χQCD is that of Ref. [7], with $N_f = 2$ massless flavors of quarks on a $8^3 \times 4$ lattice at $\gamma = 10$. In that study $T = 0$ measurements (on a $8^3 \times 24$ lattice) were only made at $6/g^2 = 5.4$. At that value of the gauge coupling they found $\langle \mathrm{Tr}\mathcal{M}_0 Q^{-1}\rangle - \frac{3}{4} = -0.0200(3)$ and $\gamma\frac{N_f}{8}\langle\sigma^2 + \pi^2\rangle = 1.0198(2)$ [17], in good agreement with Eq. (29) and the vanishing of the entropy at zero temperature. Further zero temperature measurements would be necessary to extract the energy density and pressure.

Even in the absence of such zero temperature measurements the $8^3 \times 4$ lattice data from [7] can be used with Eq. (25) to extract the entropy density. The results for the gluonic and fermionic entropy densities are plotted separately in Figs. 3 and 4, respectively. The upper lines are the tree level results and the lower lines are the one-loop corrected results. The figure shows a clear first order transition consistent with the measurements of the Wilson line and the chiral order parameter in the same study. Theoretically [1], one expects a second order transitions in the continuum theory. The first order transition here is most likely due to the small size of the lattice used. Similar results are found in conventional lattice QCD with staggered quarks on lattices of that size. Clearly, larger lattices will be necessary to determine the nature of the chiral phase transition and the equation of state of the quark-gluon plasma in the continuum limit.

IV. MEAN FIELD APPROXIMATION

In this section, the mean field treatment of the four-fermion interactions is discussed in greater detail and then used to explicitly calculate the regularization dependence of the four-fermion coupling $G = 1/\gamma$ to lowest order in the gauge coupling g. For these purposes it is best to work with the model where the $\sigma\bar{\psi}\psi$ vertex and the dynamical fermion mass are proportional to $\sqrt{G}$. This is accomplished by rescaling the auxiliary fields such that the action is

$$S = \sum_x \Big[\beta_\sigma \sum_{i<j} P_{ij}(x) + \beta_\tau \sum_j P_{0j}\Big] + \sum_{a=1}^{N_f/4}\Big\{\sum_{x,y}\bar{\chi}^a(x)[\sum_{j=1}^{3}\mathcal{M}_j(x,y) + \gamma_F \mathcal{M}_0(x,y)]\chi^a(y)$$
$$+ \sum_x \bar{\chi}^a(x)\frac{\sqrt{G}}{16}\sum_{<x,\tilde{x}>}[\sigma(\tilde{x}) + i\varepsilon(x)\pi(\tilde{x})]\chi^a(x)\Big\} + \frac{N_f}{8}\sum_{\tilde{x}}[\sigma^2(\tilde{x}) + \pi^2(\tilde{x})]. \tag{30}$$

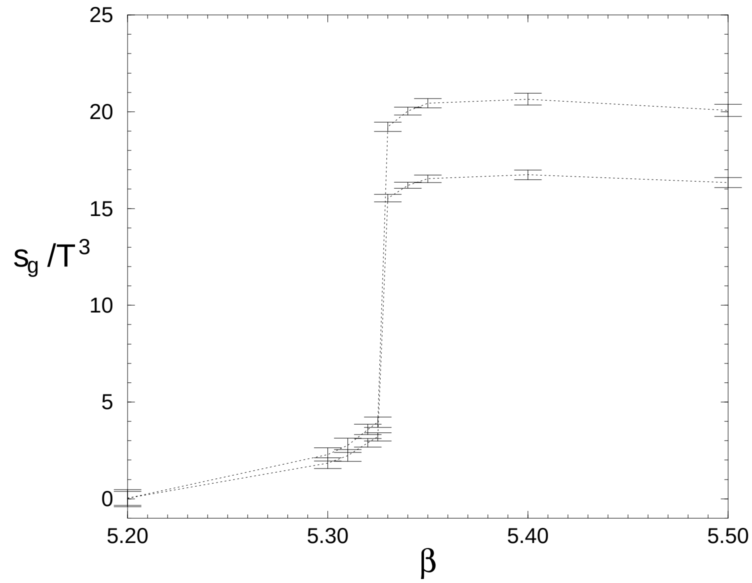


FIG. 3. Gluonic entropy density using the data from [7], for two massless flavors on a $8^3 \times 4$ lattice at $\gamma = 10$. The upper line is for the tree level results and the lower line the one-loop corrected results.

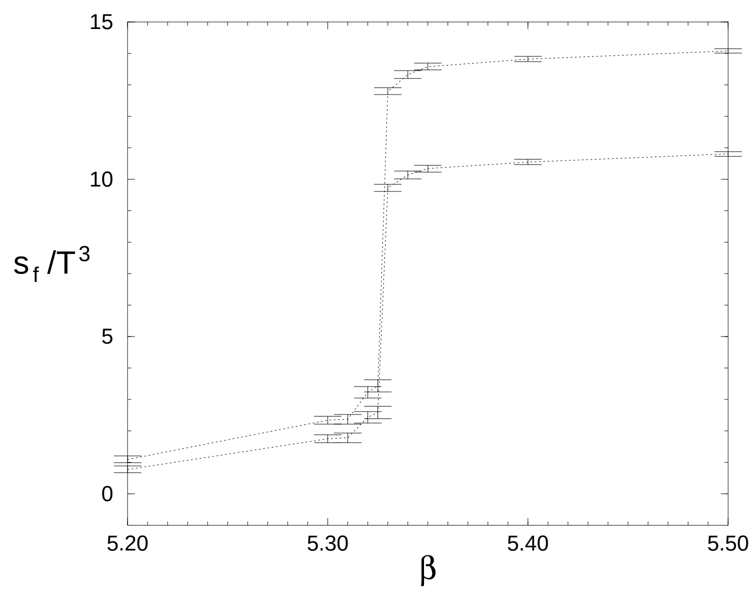


FIG. 4. Fermionic entropy density using the data from [7], for two massless flavors on a $8^3 \times 4$ lattice at $\gamma = 10$. The upper line is for the tree level results and the lower line the one-loop corrected results.

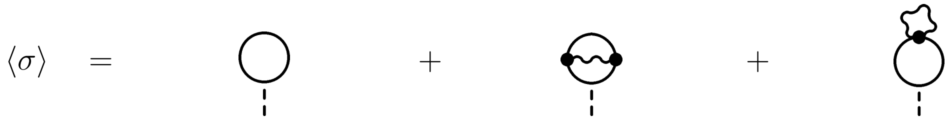


FIG. 5. Contributions to $\langle\sigma\rangle$ up through $O(g^2)$.

Using identities such as $\sqrt{G}\frac{N_f}{4}\langle\bar{\chi}(\sigma+i\varepsilon\pi)\chi\rangle = 2(1-\frac{N_f}{8}\langle\sigma^2+\pi^2\rangle)$ it is easy to verify that the final expressions of the previous section, (23) – (25), will remain unchanged.

In a mean field (or a large N_f) approximation of the four-fermion interactions the auxiliary fields take on the values

$$\langle\sigma\rangle = \sqrt{G}\langle\psi^a\bar{\psi}^a\rangle \tag{31}$$

$$= \sqrt{G}\mathrm{Tr}[S_F'(x-x)], \tag{32}$$

$$\langle\pi\rangle = \sqrt{G}\langle\psi^a\gamma_5\bar{\psi}^a\rangle \tag{33}$$

$$= \sqrt{G}\mathrm{Tr}[\gamma_5 S_F'(x-x)], \tag{34}$$

where S_F' is the full quark propagator (including the gluonic contributions):

$$S_F'^{-1}(p) = i\sum_\lambda \gamma_\lambda \sin(p_\lambda a_\lambda)/a_\lambda + \langle\sigma\rangle + i\gamma_5\tau_3\langle\pi\rangle + \Sigma(p) + O(a). \tag{35}$$

Here $\Sigma(p)$ is the quark self-energy of Sec. V B evaluated at $m^2 = \langle\sigma\rangle^2 + \langle\pi\rangle^2$. The lowest order gluonic contributions to $\langle\sigma\rangle$ are shown in Fig. 5.

Now the idea of the χQCD model is to choose the value of G to be very small, so that the breaking of the chiral symmetry is due to the *long distance* behavior of the gauge fields (as opposed to the spontaneous symmetry breaking in Gross-Neveu [18] and Nambu–Jona-Lasinio [19] type models). Since this is inherently a non-perturbative effect it will not be found in perturbation theory at any finite order of the gauge coupling. Therefore, we will have to rely on the values of $\langle\sigma\rangle$ and $\langle\pi\rangle$ being determined by Monte Carlo simulation.

For simplicity of the following discussion choose $\langle\pi\rangle = 0$. A non-vanishing value of $\langle\sigma\rangle$ then signals dynamical chiral symmetry breaking. Following the one-loop calculation of the quark propagator in Sec. V B, the dynamically generated quark mass (determined by the pole in $S_F'(p)$) is then

$$\sqrt{G}\langle\sigma\rangle\Big[1 + g^2\frac{4}{3}\Sigma_2(\xi) + O(g^4)\Big]. \tag{36}$$

Hence, the regularization dependence of this physical mass can be absorbed into the bare four-fermion coupling G by taking

$$\sqrt{G(\xi)}\langle\sigma\rangle = \sqrt{G}\langle\sigma\rangle\Big[1 + g^2 C_m(\xi) + O(g^4)\Big], \tag{37}$$

with $C_m(\xi)$ evaluated at $m = \sqrt{G}\langle\sigma\rangle$.

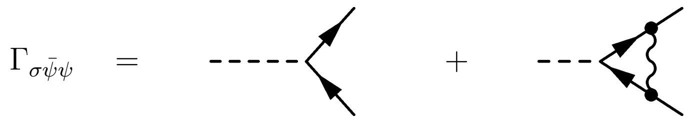


FIG. 6. Contributions to the $\sigma\bar{\psi}\psi$ vertex up through $O(g^2)$.

The ξ dependence of G can also be determined by calculating the one-loop gluonic contributions to the $\sigma\bar{\psi}\psi$ vertex shown in Fig. 6. It suffices to consider external momenta less than $1/a$. It is then easy to see that the one loop gluonic correction to $\Gamma_{\sigma\bar{\psi}\psi}$ is identical to the scalar part of the one loop fermion self energy calculated in Sec. V B:

$$\Gamma_{\sigma\bar{\psi}\psi} = \frac{\sqrt{G}}{N_f}\Big[1 - g^2\frac{N^2-1}{2N}\sum_{\mu}\int_q \gamma_\mu S_F(p+q)S_F(p+k+q)\gamma_\mu D(q) \times \cos((p_\mu + q_\mu/2)a_\mu)\cos((p_\mu + k_\mu + q_\mu/2)a_\mu)\Big] \tag{38}$$

$$= \frac{\sqrt{G}}{N_f}\Big[1 + g^2\frac{4}{3}\int_q \frac{4 - \frac{1}{2}\Delta_1(1)}{2\Delta_1(\xi)(\Delta_2(\xi) + G\langle\sigma\rangle^2)/a^4} + \text{terms higher O(pa, ka)}\Big] \tag{39}$$

$$= \frac{\sqrt{G}}{N_f}\Big[1 + g^2\frac{4}{3}\Sigma_2(\xi)\Big], \tag{40}$$

where Σ_2 is the same as for ordinary QCD [Eq. (73)] with $m = \sqrt{G}\langle\sigma\rangle$. Defining $\Gamma_{\sigma\bar{\psi}\psi}(\xi) = Z_\psi^{-1}(\xi)\Gamma^{sym}_{\sigma\bar{\psi}\psi}$, the ξ dependence is absorbed by taking $\sqrt{G(\xi)} = \sqrt{G}[1+g^2\frac{4}{3}(\Sigma_{1,\sigma}(\xi) - \Sigma_2(\xi))] = \sqrt{G}[1 + g^2 C_m(\xi)]$, in agreement with Eq. (37).

Differentiating this with respect to ξ yields

$$\frac{1}{G}\Big(\frac{\partial G}{\partial \xi}\Big)_{\xi=1} = 2g^2\Big(\frac{\partial C_m}{\partial \xi}\Big)_{\xi=1}. \tag{41}$$

The scaling of G with the spatial lattice spacing (at fixed ξ) is then given by $G(a) = G[1 + g^2\frac{8}{3}(\Sigma_{1,\sigma} - \Sigma_2)_{DIV}]$, where $(\Sigma_{1,\sigma})_{DIV} = -1/(8\pi^2)\ln(a)$ and $(\Sigma_2)_{DIV} = -1/(2\pi^2)\ln(a)$ are the divergent parts of $\Sigma(p)$ as $a \to 0$ [20]. This yields

$$\frac{a}{G}\frac{\partial G}{\partial a} = g^2\frac{2}{2\pi^2} + O(g^4). \tag{42}$$

V. LATTICE QCD THERMODYNAMICS WITH NONZERO QUARK MASSES

This section briefly reviews the perturbative thermodynamics of lattice QCD, including the contributions due to nonzero bare quark masses. The necessary one-loop calculations of [11–13] are then extended to nonzero quark mass and the results discussed.

On an anisotropic lattice the action for QCD with N_f (degenerate) flavors of staggered fermions is

$$S = \sum_x \Big[\beta_\sigma \sum_{i<j} P_{ij}(x) + \beta_\tau \sum_j P_{0j}\Big] + \sum_{a=1}^{N_f/4} \sum_{x,y} a^2 a_\tau \bar{\chi}^a(x) Q(x,y) \chi^a(y), \tag{43}$$

with

$$Q(x,y) = \sum_{j=1}^{3} \mathcal{M}_j(x,y) + \gamma_F \mathcal{M}_0(x,y) + \delta_{x,y} m_0 a. \tag{44}$$

The separate gauge couplings $\beta_\sigma = 6/g_\sigma^2$, $\beta_\tau = 6/g_\tau^2$, as well as the extra parameter γ_F, are necessary to maintain Euclidean invariance (i. e. regularization independence) in the continuum limit. The functional dependence of these parameters on $\xi \equiv a/a_\tau$ is fixed by requiring the theory to be independent of ξ in the continuum limit.

As shown in Sec. III, to calculate thermodynamic quantities one needs

$$\begin{aligned}\left\langle \xi \frac{\partial S}{\partial \xi} \right\rangle = N_\sigma^3 N_\tau \Big[& \xi \frac{\partial \beta_\sigma}{\partial \xi} \langle P_{ss} \rangle + \xi \frac{\partial \beta_\tau}{\partial \xi} \langle P_{st} \rangle \\ & + \xi \frac{\partial \gamma_F}{\partial \xi} \frac{N_f}{4} \langle \bar{\chi} \mathcal{M}_0 \chi \rangle + \xi \frac{\partial m_0 a}{\partial \xi} \frac{N_f}{4} \langle \bar{\chi}\chi \rangle + \Big(-1 + \xi \frac{\partial Z_\psi}{\partial \xi}\Big) \frac{N_f}{4} \langle \bar{\chi} Q \chi \rangle \Big],\end{aligned} \tag{45}$$

and

$$\begin{aligned}\left\langle a \frac{\partial S}{\partial a} \right\rangle = N_\sigma^3 N_\tau \Big[& a \frac{\partial \beta_\sigma}{\partial a} \langle P_{ss} \rangle + a \frac{\partial \beta_\tau}{\partial a} \langle P_{st} \rangle \\ & + a \frac{\partial \gamma_F}{\partial a} \frac{N_f}{4} \langle \bar{\chi} \mathcal{M}_0 \chi \rangle + a \frac{\partial (m_0 a)}{\partial a} \frac{N_f}{4} \langle \bar{\chi}\chi \rangle + \Big(3 + \frac{\partial Z_\psi}{\partial \ln a}\Big) \frac{N_f}{4} \langle \bar{\chi} Q \chi \rangle \Big].\end{aligned} \tag{46}$$

The averages appearing above can be evaluated by Monte Carlo simulation. $\langle \bar{\chi} Q \chi \rangle = -\mathrm{Tr}(QQ^{-1}) = -3$, for 3 colors of quarks. Hence, when subtracting $T = 0$ measurements on symmetric lattices the last terms appearing in Eqs. (45) and (46) will not contribute. In general, the various derivatives of the coefficients may also need to be calculated non-perturbatively [21]. As a first step, however, they can be estimated by lattice perturbation theory.

Considering first the ξ dependence:

$$\beta_\sigma(\xi) = \frac{6}{\xi g^2}\Big[1 + C_\sigma(\xi) g^2 + O(g^4)\Big], \tag{47}$$

$$\beta_\tau(\xi) = \frac{6\xi}{g^2}\Big[1 + C_\tau(\xi) g^2 + O(g^4)\Big], \tag{48}$$

$$\gamma_F(\xi) = \xi\Big[1 + C_F(\xi) g^2 + O(g^4)\Big], \tag{49}$$

$$m_0(\xi) = m_0\Big[1 + C_m(\xi) g^2 + O(g^4)\Big]. \tag{50}$$

[Here $g \equiv g_{\sigma,\tau}(\xi = 1)$ and $m_0 \equiv m_0(\xi = 1)$.] The coefficients C_σ, C_τ, C_F and C_m all vanish at $\xi = 1$ where most Monte Carlo simulations are performed. However, their derivatives with respect to ξ are nonzero at $\xi = 1$. They are calculated in Secs. V A and V B.

The implicit scaling of the gauge coupling g and the bare quark mass m_0 with the spatial lattice spacing is governed by the renormalization group equations:

$$
\begin{aligned}
-a\frac{\partial g}{\partial a} &= \beta(g) = -\beta_0 g^2 - \beta_1 g^5 + O(g^7), &(51)\\
\beta_0 &= \frac{11}{16\pi^2}\Big(1-\frac{2N_f}{33}\Big), \qquad \beta_1 = \frac{1}{16\pi^2}\Big[102-\Big(10+\frac{8}{3}\Big)N_f\Big], &(52)\\
a\frac{\partial m_0}{\partial a} &= m_0\gamma(g) = m_0\Big[\gamma_0 g^2 + O(g^4)\Big], &(53)\\
\gamma_0 &= \frac{1}{2\pi^2}. &(54)
\end{aligned}
$$

$\beta(g)$ is universal (regularization independent) up through $O(g^5)$, and $\gamma(g)$ is universal through $O(g^2)$. At $\xi = 1$

$$
a\frac{\partial \beta_\sigma}{\partial a} = a\frac{\partial \beta_\tau}{\partial a} = a\frac{\partial (6/g^2)}{\partial a} = -12\beta_0 + O(g^2), \tag{55}
$$

and

$$
a\frac{\partial \gamma_F}{\partial a} = 0. \tag{56}
$$

The final perturbative expressions for the energy, pressure, and entropy (at $\xi = 1$) are:

$$
\begin{aligned}
\epsilon = T^4 N_\tau^4 \Big\{ & \frac{6}{g^2}\Big(1 - g^2\frac{\partial C_\sigma}{\partial \xi}\Big)\langle P_{ss}\rangle + \frac{6}{g^2}\Big(-1 - g^2\frac{\partial C_\tau}{\partial \xi}\Big)\langle P_{st}\rangle \\
& + \Big(1 + g^2\frac{\partial C_F}{\partial \xi}\Big)\frac{N_f}{4}\langle \mathrm{Tr}\mathcal{M}_0 Q^{-1}\rangle \\
& + m_0 a g^2 \frac{\partial C_m}{\partial \xi}\frac{N_f}{4}\langle \mathrm{Tr} Q^{-1}\rangle + 3\frac{N_f}{4}\Big(-1 + \frac{\partial Z_\psi}{\partial \xi}\Big)\Big\}, &(57)\\
p = \frac{\epsilon}{3} + \frac{T^4}{3}N_\tau^4 \Big\{ & 12\beta_0\Big(\langle P_{ss}\rangle + \langle P_{st}\rangle\Big) \\
& + m_0 a(1 + g^2\gamma_0)\frac{N_f}{4}\langle \mathrm{Tr} Q^{-1}\rangle + 3\frac{N_f}{4}\Big(3 + \frac{\partial Z_\psi}{\partial \ln a}\Big)\Big\}, &(58)\\
s = \frac{4T^3}{3}N_\tau^4 \Big\{ & \frac{6}{g^2}\Big(1 - g^2\frac{\partial C_\sigma}{\partial \xi}\Big)\langle P_{ss}\rangle + \frac{6}{g^2}\Big(-1 - g^2\frac{\partial C_\tau}{\partial \xi}\Big)\langle P_{st}\rangle + 3\beta_0\Big(\langle P_{ss}\rangle + \langle P_{st}\rangle\Big) \\
& + \Big(1 + g^2\frac{\partial C_F}{\partial \xi}\Big)\frac{N_f}{4}\langle \mathrm{Tr}\mathcal{M}_0 Q^{-1}\rangle - \frac{3}{4}\frac{N_f}{4}\Big(1 - g^2 4\frac{\partial Z_\psi}{\partial \xi} - g^2\frac{\partial Z_\psi}{\partial \ln a}\Big) \\
& + m_0 a\Big(1 + g^2\gamma_0 + g^2 4\frac{\partial C_m}{\partial \xi}\Big)\frac{N_f}{16}\langle \mathrm{Tr} Q^{-1}\rangle\Big\} &(59)\\
= \frac{4T^3}{3}N_\tau^4 \Big\{ & \frac{6}{g^2}\Big[1 + g^2\frac{1}{2}\Big(\frac{\partial C_\tau}{\partial \xi} - \frac{\partial C_\sigma}{\partial \xi}\Big)\Big]\Big[\langle P_{ss}\rangle - \langle P_{st}\rangle\Big] \\
& + \Big(1 + g^2\frac{\partial C_F}{\partial \xi}\Big)\frac{N_f}{4}\Big[\langle \mathrm{Tr}\mathcal{M}_0 Q^{-1}\rangle - \frac{3}{4} + \frac{1}{4}m_0 a\langle \mathrm{Tr} Q^{-1}\rangle\Big]\Big\}. &(60)
\end{aligned}
$$

The $m_0 \approx 0$ relations given by Eqs. (66), (82) and (83) have been used to simplify the expression for the entropy density. In most cases one is interested in measuring the energy density and pressure relative to the $T = 0$ vacuum, by subtracting measurements on symmetric lattices. In which case, the last terms in Eqs. (57) and (58) will drop out. Using

$\langle \mathrm{Tr} QQ^{-1} \rangle = 3$ it follows that $\langle \mathrm{Tr}\mathcal{M}_0 Q^{-1} \rangle_{sym} = \frac{3}{4} - \frac{1}{4} m_0 a \langle \mathrm{Tr} Q^{-1} \rangle_{sym}$. Hence, it is easy to verify that the expression for the entropy vanishes on an isotropic, symmetric lattice.

The one loop perturbative calculations of the various coefficients appearing above, and in particular their dependence on the bare quark mass m_0, are presented in the following two sections.

A. Vacuum Polarization

Define $C_\sigma(\xi) \equiv C_\sigma^G(\xi) + C_\sigma^F(\xi)$ and $C_\tau(\xi) \equiv C_\tau^G(\xi) + C_\tau^F(\xi)$ where the superscripts G and F refer to the gauge and fermionic contributions to the QCD vacuum polarization, respectively. The ξ dependence is most easily calculated using a background field method [14,11]; i. e. by expanding the gauge action around a classical background gauge configuration and requiring the effective action to be independent of ξ.

The pure gauge contributions have been calculated by Karsch [11]. They yield $(\partial C_\sigma^G/\partial\xi)_{\xi=1} = 0.20161$ and $(\partial C_\tau^G/\partial\xi)_{\xi=1} = -0.13195$. The fermionic contributions are given solely by the one-loop vacuum polarization graphs of Eqs. (7) and (8). These have been evaluated previously by Trinchero for massless quarks [12]. Following his notation,

$$\Pi_{ij} = (-k_i k_j I_\sigma + \delta_{ij})(\mathbf{k^2} I_\sigma + k_0^2 I_\tau), \tag{61}$$

$$\Pi_{\mu 0} = (-k_\mu k_0 + \delta_{\mu 0} k^2) I_\tau, \tag{62}$$

with

$$I_\sigma(\xi) = -\frac{N_f}{2} \int_{q/2} \frac{\cos^2(q_1 a)\cos^2(q_2 a) - \frac{1}{3}\cos(2q_1 a)\cos(2q_2 a)}{[\Delta_2(\xi) + m_0^2 a^2]^2/a^4}, \tag{63}$$

$$I_\tau(\xi) = -\frac{N_f}{2} \int_{q/2} \frac{\cos^2(q_1 a)\cos^2(q_0 a_\tau) - \frac{1}{3}\cos(2q_1 a)\cos(2q_0 a_\tau)}{[\Delta_2(\xi) + m_0^2 a^2]^2/a^4}. \tag{64}$$

C_σ^F and C_τ^F are related to these integrals by

$$C_\sigma^F = 2[I_\sigma(\xi) - I_\sigma(1)], \qquad C_\tau^F = 2[I_\tau(\xi) - I_\tau(1)]. \tag{65}$$

For $m_0 = 0$ I find $(\partial C_\sigma^F/\partial\xi)_{\xi=1} = -\frac{N_f}{2} 0.000622$ and $(\partial C_\tau^F/\partial\xi)_{\xi=1} = -\frac{N_f}{2} 0.007822$, in agreement with [12]. The behaviors of $(\partial C_\sigma^F/\partial\xi)_{\xi=1}$ and $(\partial C_\tau^F/\partial\xi)_{\xi=1}$ (for $N_f = 2$) as a function of $m_0 a$ are shown in Figs. 7 and 8, respectively. As $m_0 a$ increases from zero to 0.1, $\partial C_\sigma/\partial\xi = \partial C_\sigma^G/\partial\xi + \partial C_\sigma^F/\partial\xi$ decreases 0.1% and $\partial C_\tau/\partial\xi = \partial C_\tau^G/\partial\xi + \partial C_\tau^F/\partial\xi$ increases 0.05% (per $N_f/2$ flavors). Therefore, in most cases when $m_0 a$ is small these deviations can be ignored and the massless quark values used.

The invariance of the string tension [11] as well as the vanishing of the gluonic entropy at zero temperature requires

$$\left(\frac{\partial \beta_\sigma}{\partial \xi}\right)_{\xi=1} + \left(\frac{\partial \beta_\tau}{\partial \xi}\right)_{\xi=1} = -\frac{1}{2}\frac{\partial(6/g^2)}{\partial \ln a}. \tag{66}$$

To lowest order in g this reads $(\partial C_\sigma/\partial\xi)_{\xi=1} + (\partial C_\tau/\partial\xi)_{\xi=1} = \beta_0$. This relation no longer holds at nonzero $m_0 a$. However, given the relatively small variations in $(\partial C_\sigma/\partial\xi)_{\xi=1}$ and $(\partial C_\tau/\partial\xi)_{\xi=1}$ it is still effectively satisfied for small bare masses.

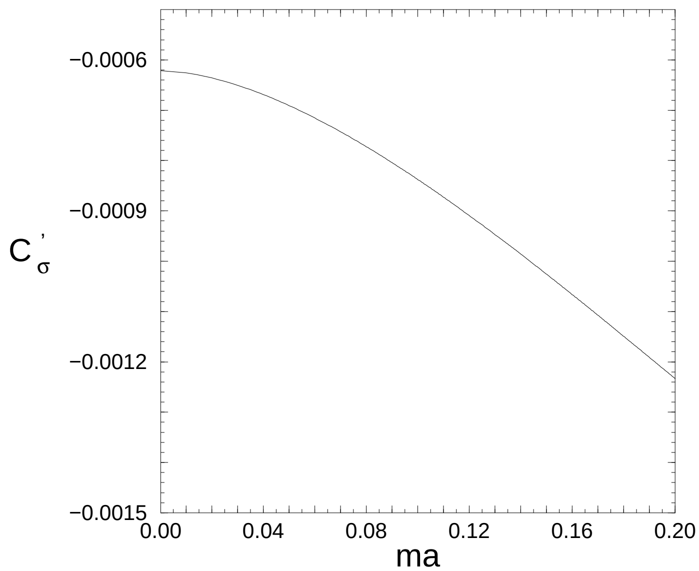


FIG. 7. $(\partial C_\sigma^F/\partial\xi)_{\xi=1}$ as a function of the mass (for two quark flavors).

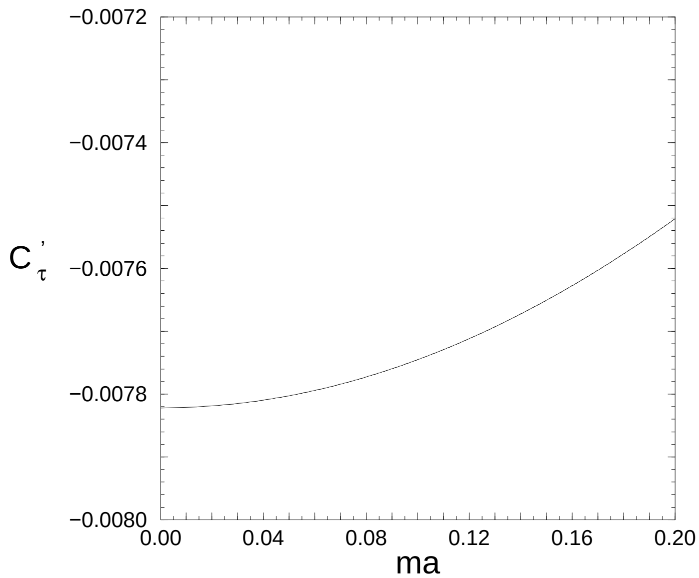


FIG. 8. $(\partial C_\tau^F/\partial\xi)_{\xi=1}$ as a function of the mass (for two quark flavors).

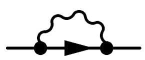
B. Quark Propagator

$\gamma_F(\xi)$ can be found by calculating the one loop corrections to the fermion propagator (i.e. the fermionic self-energy) and then demanding rotational invariance of the propagator in the continuum limit. This has also been carried out by Karsch for massless fermions [13], and will be extended here to the more general case.

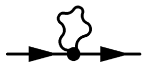
Consider

$$\Gamma^{(2)}(p) \equiv S_F^{-1}(p) + \Sigma(p). \tag{67}$$

To lowest order in g, $-\Sigma(p)$ is given by the sum of the two terms:



$$= -g^2 \frac{N^2-1}{2N} \sum_\mu \int_q \gamma_\mu S_F(p+q)\gamma_\mu D(q)\cos^2(p_\mu a_\mu + q_\mu a_\mu/2), \tag{68}$$



$$= g^2 \frac{N^2-1}{2N} \frac{1}{2} \sum_\mu a_\mu i\gamma_\mu \sin(p_\mu a_\mu) \int_q D(q). \tag{69}$$

Expanding the self energy in powers of p_ν,

$$\Sigma(p) = g^2 \frac{N^2-1}{2N} \Big[\sum_j i\gamma_j p_j \Sigma_{1,\sigma} + i\gamma_0 p_0 \Sigma_{1,\tau} + m_0 \Sigma_2 \Big] + \text{terms of higher O(pa)}, \tag{70}$$

with

$$\Sigma_{1,\sigma} = \int_q \Big\{ \frac{a^4 \sin^2(q_1 a)}{2\Delta_1(\xi)(\Delta_2(\xi) + m_0^2 a^2)} \Big[\frac{1}{2} + \frac{3 - \cos(q_1 a) - \frac{1}{2}\Delta_1(1)}{\Delta_1(\xi)} \Big] - \frac{a^4}{4\Delta_1(\xi)} \Big\}, \tag{71}$$

$$\Sigma_{1,\tau} = \int_q \Big\{ \frac{a^4 \sin^2(q_0 a_\tau)}{2\Delta_1(\xi)(\Delta_2(\xi) + m_0^2 a^2)} \Big[\frac{1}{2} + \xi^2 \frac{3 - \cos(q_0 a_\tau) - \frac{1}{2}\Delta_1(1)}{\Delta_1(\xi)} \Big] - \frac{a^4}{4\xi^2 \Delta_1(\xi)} \Big\}, \tag{72}$$

and

$$\Sigma_2 = \int_q \frac{4 - \frac{1}{2}\Delta_1(1)}{2\Delta_1(\xi)(\Delta_2(\xi) + m_0^2 a^2)/a^4}. \tag{73}$$

Hence,

$$\begin{aligned}\Gamma^{(2)} = \sum_j i\gamma_j p_j \Big[1 + g^2 \frac{N^2-1}{2N} \Sigma_{1,\sigma}(\xi) \Big] + i\gamma_0 p_0 \Big[\gamma_F/\xi + g^2 \frac{N^2-1}{2N} \Sigma_{1,\tau}(\xi) \Big] \\ + m_0 \Big[1 + g^2 \frac{N^2-1}{2N} \Sigma_2(\xi) \Big].\end{aligned} \tag{74}$$

Demanding rotational invariance of $\Gamma^{(2)}$ in the continuum limit yields

$$\gamma_F(\xi) = \xi[1 + g^2 C_F(\xi) + O(g^4)], \tag{75}$$

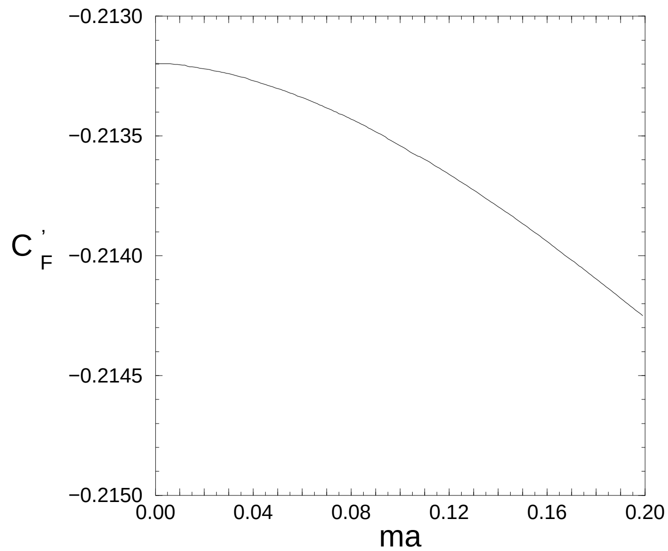


FIG. 9. $(\partial C_F/\partial\xi)_{\xi=1}$ as a function of the mass.

with

$$C_F(\xi) = \frac{N^2-1}{2N}\Big(\Sigma_{1,\sigma}(\xi) - \Sigma_{1,\tau}(\xi)\Big). \tag{76}$$

For $m_0 = 0$ I find $(\partial C_F/\partial\xi)_{\xi=1} = -0.2132$, in agreement with [13]. For nonzero quark masses $(\partial C_F/\partial\xi)_{\xi=1}$ is shown in Fig. 9. It decreases less than 0.2% as $m_0 a$ varies from zero to 0.1. Thus, in most cases the $m_0 = 0$ value can be safely used.

The remaining ξ dependence of $\Gamma^{(2)}$ can be absorbed by a wavefunction rescaling:

$$\Gamma^{(2)}(\xi) = Z_\psi^{-1}(\xi)\Gamma^{(2)}_{sym}, \tag{77}$$

$$Z_\psi(\xi) = 1 - g^2\frac{N^2-1}{2N}\Sigma_{1,\sigma}(\xi), \tag{78}$$

and a functional bare mass dependence

$$m_0(\xi) = m_0\Big[1 + g^2 C_m(\xi) + O(g^4)\Big], \tag{79}$$

$$C_m(\xi) = \frac{N^2-1}{2N}\Big(\Sigma_{1,\sigma}(\xi) - \Sigma_2(\xi)\Big). \tag{80}$$

At $m_0 = 0$ $(\partial\Sigma_{1,\sigma}/\partial\xi)_{\xi=1} = -0.0368$ and decreases just 0.2% as m_0 increases to 0.1. The divergent part of $\Sigma_{1,\sigma}$ at $\xi = 1$ is $[\Sigma_{1,\sigma}(1)]_{DIV} = -1/(8\pi^2)\ln(a)$ [20]. Thus, $\partial Z_\psi/\partial\ln a = g^2/(6\pi^2)$.

Turning to the mass,

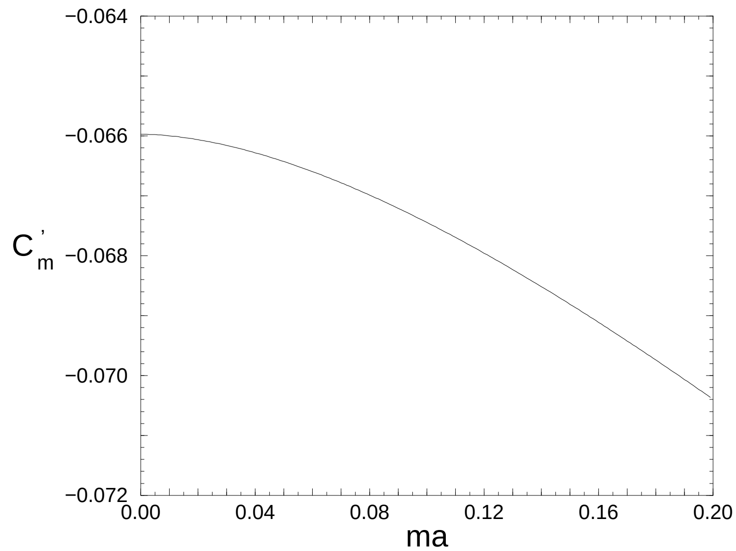


FIG. 10. $(\partial C_m/\partial\xi)_{\xi=1}$ as a function of the mass.

$$\left(\frac{\partial m_0}{\partial\xi}\right)_{\xi=1} = m_0 g^2\left(\frac{\partial C_m}{\partial\xi}\right)_{\xi=1}. \tag{81}$$

$(\partial C_m/\partial\xi)_{\xi=1} = -0.066$ at $m_0a = 0$. Figure 10 shows the behavior of $(\partial C_m/\partial\xi)_{\xi=1}$ as a function of m_0a. Its value decreases about 2% as m_0a increases from zero to 0.1. This is significantly more variation than seen in $\partial C_F/\partial\xi$, $\partial C_\sigma/\partial\xi$ and $\partial C_\tau/\partial\xi$, and is due mainly to the variation in $\partial\Sigma_2/\partial\xi$. Values of $(\partial C_m/\partial\xi)_{\xi=1}$ are also given in Table I at selected values of m_0a.

Requiring the entropy to vanish on an isotropic, symmetric lattice yields

$$g^2\left(\frac{\partial C_F}{\partial\xi}\right)_{\xi=1} + 4\left(\frac{\partial Z_\psi}{\partial\xi}\right)_{\xi=1} + \frac{\partial Z_\psi}{\partial\ln a} = 0 \tag{82}$$

and

$$-g^2\left(\frac{\partial C_F}{\partial\xi}\right)_{\xi=1} + g^2 4\left(\frac{\partial C_m}{\partial\xi}\right)_{\xi=1} + \frac{\partial\ln m_0}{\partial\ln a} = 0, \tag{83}$$

which are satisfied by the numerical values given for $m_0a \approx 0$. Like the corresponding sum rule for the gauge couplings (66) they are slightly violated at larger m_0a.

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APPENDIX

On an anisotropic lattice with $\xi \equiv a/a_\tau$ the free ($g = 0$) lattice propagators are

$$D(k) = \frac{1}{\sum_\lambda [1 - \cos(k_\lambda a_\lambda)]/a_\lambda^2} = \frac{1}{2\Delta_1(\xi)/a^2}, \tag{84}$$

$$S_F(p) = \frac{1}{i\sum_\lambda \gamma_\lambda \sin(p_\lambda a_\lambda)/a_\lambda + m} = \frac{-i\sum_\lambda \gamma_\lambda \sin(p_\lambda)/a_\lambda + m}{\Delta_2(\xi)/a^2 + m^2}, \tag{85}$$

where

$$\Delta_1(\xi) \equiv \sum_{1=1}^{3} (1 - \cos(k_i a)) + \xi^2 (1 - \cos(k_0 a_\tau)), \tag{86}$$

$$\Delta_2(\xi) \equiv \sum_{1=1}^{3} \sin^2(p_i a) + \xi^2 \sin^2(p_0 a_\tau). \tag{87}$$

Integration limits have been denoted by

$$\int_q \equiv \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 q}{(2\pi)^4}, \qquad\qquad \int_{q/2} \equiv \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4 q}{(2\pi)^4}. \tag{88}$$

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ma	C'_m	ma	C'_m
0.000	-0.0660	0.100	-0.0674
0.005	-0.0660	0.105	-0.0676
0.010	-0.0660	0.110	-0.0677
0.015	-0.0660	0.115	-0.0678
0.020	-0.0661	0.120	-0.0680
0.025	-0.0661	0.125	-0.0681
0.030	-0.0662	0.130	-0.0682
0.035	-0.0662	0.135	-0.0684
0.040	-0.0663	0.140	-0.0685
0.045	-0.0664	0.145	-0.0687
0.050	-0.0664	0.150	-0.0688
0.055	-0.0665	0.155	-0.0690
0.060	-0.0666	0.160	-0.0691
0.065	-0.0667	0.165	-0.0693
0.070	-0.0668	0.170	-0.0694
0.075	-0.0669	0.175	-0.0696
0.080	-0.0670	0.180	-0.0697
0.085	-0.0671	0.185	-0.0699
0.090	-0.0672	0.190	-0.0701
0.095	-0.0673	0.195	-0.0702

TABLE I. $(\partial C_m/\partial\xi)_{\xi=1}$ at selected values of ma.